FOLIA 163

Annales Universitatis Paedagogicae Cracoviensis

Studia ad Didacticam Mathematicae Pertinentia VI (2014)

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Fibonacci polynominals of order k*

Abstract. In this paper we give formulas determining the Fibonacci polynomials of order k using the so-called generalized Newton symbols, i.e., the coefficients in the expansion of $(1 + z + z^2 + \ldots + z^{k-1})^n$ with respect to the powers of z.

The Fibonacci polynomials of order 3 and 4 were introduced by V. E. Hoggatt and M. Bicknel in 1973 in (Hoggatt, Bicknell, 1973). Ten years later A. N. Philippou, C. Georghim and G. N. Philippou generalized this notion for an arbitrary integer order $k \geqslant 2$ and obtained expansions of the Fibonacci polynomials in terms of the multinomial and binomial coefficients, respectively (see Philippou, Georghiou, Philippou, 1983).

In this paper we derive formulas determining the Fibonacci polynomials of order k using the polynomial coefficients introduced by André (1876), also called the generalized Newton symbols. These formulas seem to be much simpler than the ones given in (Philippou et al., 1983).

It is worth pointing out that the sequence of Fibonacci polynomials might be considered as an example of the generalization of a mathematical concept. Namely, it generalizes the notions of the sequence of Fibonacci numbers and the sequence of Pell numbers.

In the sequel, $\mathbb{N}$ stands for the set of non-negative integers, $\mathbb{N}_m := \{m, m+1, \ldots\}$ for $m \in \mathbb{N}$ such that $m > 0$ and by $\binom{n,k}{j}$, $j, n \in \mathbb{N}$, $k \in \mathbb{N}_2$ we denote the coefficients in the following expansion

$$(1 + z + z^2 + \ldots + z^{k-1})^n = \sum_{j=0}^{\infty} \binom{n,k}{j} z^j. \tag{1}$$

Obviously, $\binom{n,k}{j} = 0$ for $j > (k–1)n$. For simplicity of notation we set $\binom{n,k}{j} := 0$ for $n, j \in \mathbb{Z}$, $j, n < 0$ and $k \in \mathbb{N}_2$. Thus the symbol $\binom{n,k}{j}$ is defined for all $n, j \in \mathbb{Z}$ and $k \in \mathbb{N}_2$.

Let us notice that $\binom{n,2}{j} = \binom{n}{j}$, $n \in \mathbb{Z}$, $j \in \mathbb{Z}$.

*Wielomiany Fibonacciego stopnia k

2010 Mathematics Subject Classification: Primary: 11B39, 11B65

Key words and phrases: Fibonacci polynominals, Fibonacci numbers, Pell numbers, polynomial coefficients

Many other properties of the generalized Newton symbols may be found in (Belbachir, Bouroubi, Khelladi, 2008; Górowski, Łomnicki, 2010). Here we recall some of them

Lemma 1 ((Belbachir et al., 2008), (Górowski, Łomnicki, 2010))
For every $n, m \in \mathbb{N}$ and $k \in \mathbb{N}_2$,

(i) $\binom{1,k}{m} = 1$ *for* $m \in \{0, 1, 2, \ldots, (k-1)\}$,

(ii) $\binom{n+1,k}{m} = \sum_{j=0}^{k-1} \binom{n,k}{m-j}$ *for* $m \in \{0, 1, 2, \ldots, (k-1)(n+1)\}$,

(iii) $\binom{n,k}{m} = \binom{n+m-1}{m}$ *for* $m \leqslant k-1$.

(iv) $\binom{n,k}{m} = \sum_{j \geqslant 0} (-1)^j \binom{n}{j} \binom{m-jk+n-1}{n-1}$ *for* $m \in \mathbb{N}$.

The original definition of the Fibonacci polynomials might be found in (Philippou et al., 1983).

Definition 1
Let $k \in \mathbb{N}_2$, a sequence of polynomials $(F_0^k(x), F_1^k(x), F_2^k(x), \ldots)$ is called a sequence of the Fibonacci polynomials of order k if

$$\begin{cases} F_n^k(x) = n & \text{for } n \in \{0,1\}, \\ F_n^k(x) = \sum_{i=1}^{n} x^{k-i} F_{n-i}^k(x) & \text{for } n \in \{2, 3, \ldots, k\}, \\ F_n^k(x) = \sum_{i=1}^{k} x^{k-i} F_{n-i}^k(x) & \text{for } n \geqslant k+1. \end{cases} \tag{2}$$

The consecutive terms of the above sequence are called the Fibonacci polynomials of order k. Notice that $(F_n^k(1))_{n\in\mathbb{N}}$ and $(F_n^k(2))_{n\in\mathbb{N}}$ are the sequences of Fibonacci and Pell numbers of order k, respectively.

Observe that equalities (2) are equivalent to

$$\begin{cases} F_n^k(x) = n & \text{for } n \in \{0,1\}, \\ F_n^k(x) = \sum_{i=1}^{n} x^{k-i} F_{n-i}^k(x) & \text{for } n \in \{2, 3, \ldots, k-1\}, \\ F_n^k(x) = \sum_{i=1}^{k} x^{k-i} F_{n-i}^k(x) & \text{for } n \geqslant k. \end{cases} \tag{3}$$

Moreover, we show that

Lemma 2
Let $k \in \mathbb{N}_2$ and $n \in \{2, 3, \ldots, k-1\}$, then

$$F_n^k(x) = \sum_{j=0}^{n-2} \binom{n-2}{j} x^{(k-1)(n-1)-kj}, \tag{4}$$

where $F_n^k(x)$ for $n \in \{2, 3, \ldots, k-1\}$ are the Fibonacci polynomials of order k.

Proof. We will prove (4) by induction on n. Fix $k \in \mathbb{N}_2$ and observe that for $n = 2$, by (2), we have $F_2^k(x) = x^{k-1}$. The right-hand side of (4) is equal to x^{k-1}, thus (4) holds for $n = 2$.

Now suppose that (4) is true for arbitrary $2 \leqslant n \leqslant k-2$. By definition (2) we have

$$F_{n+1}^k(x) = \sum_{i=1}^{n+1} x^{k-i} F_{n+1-i}^k(x).$$

Hence, in view of the fact that $F_0^k(x) = 0$ and from the induction hypothesis we obtain

$$\begin{aligned}
\sum_{i=1}^{n+1} x^{k-i} F_{n+1-i}^k(x) &= \sum_{i=0}^{n-1} x^{k-i-1} F_{n-i}^k(x) = \sum_{i=0}^{n-2} x^{k-i-1} F_{n-i}^k(x) + x^{k-n} \\
&= \sum_{i=0}^{n-2} x^{k-i-1} \sum_{j=0}^{n-i-2} \binom{n-i-2}{j} x^{(k-1)(n-i-1)-kj} + x^{k-n} \\
&= \sum_{i=0}^{n-2} x^{k-i-1+(k-1)(n-i-1)} \sum_{j=0}^{n-i-2} \binom{n-i-2}{j} (x^{-k})^j + x^{k-n} \\
&= \sum_{i=0}^{n-2} x^{(k-1)(n-i)-i} (1 + x^{-k})^{n-i-2} + x^{k-n} \\
&= \sum_{i=0}^{n-2} x^{2k-n} (1 + x^k)^{n-2} \left(\frac{1}{1+x^k} \right)^i + x^{k-n} \\
&= x^{2k-n} (1 + x^k)^{n-2} \frac{1 - \left(\frac{1}{1+x^k} \right)^{n-1}}{1 - \frac{1}{1+x^k}} + x^{k-n} \\
&= x^{k-n} (1 + x^k)^{n-1} \left(1 - \left(\frac{1}{1+x^k} \right)^{n-1} \right) + x^{k-n} \\
&= x^{k-n} (1 + x^k)^{n-1} = x^{k-n} \sum_{j=0}^{n-1} \binom{n-1}{j} x^{k(n-j-1)} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} x^{(k-1)n-kj}
\end{aligned}$$

and the proof of the induction step is complete. Thus by the principle of induction equality (4) holds.

Our main result is

Theorem 1
For any $k \in \mathbb{N}_2$ the Fibonacci polynomials of order k can be expressed as

$$F_n^k(x) = \sum_{j \geqslant 0} \binom{n-1-j, k}{j} x^{(k-1)(n-1)-kj}, \quad n \in \mathbb{N}. \tag{5}$$

Proof. Observe that from (5), in view of the fact that $\binom{0,k}{0} = 1$, $k \in \mathbb{N}_2$, we have

$$F_0^k(x) = 0 \quad \text{and} \quad F_1^k(x) = \binom{0, \mu}{0} x^0 = 1$$

for every $k \in \mathbb{N}_2$.

Fix arbitrary $k \in \mathbb{N}_2$. For $n \in \{2, 3, \ldots, k-1\}$ equality (iii) from Lemma 1 yields

$$F_n^k(x) = \sum_{j \geqslant 0} \binom{n-1-j, k}{j} x^{(k-1)(n-1)-kj} = \sum_{j \geqslant 0} \binom{n-2}{j} x^{(k-1)(n-1)-kj}.$$

Thus by Lemma 2, $F_n^k(x)$, $n \in \{2, 3, \ldots, k-1\}$ are the Fibonacci polynomials of order k.

Finally, assume that $n \geqslant k$. By Lemma 1 (ii) and (5) we obtain

$$\begin{aligned}
\sum_{i=1}^{k} x^{k-i} F_{n-i}^k(x) &= \sum_{i=1}^{k} \sum_{j \geqslant 0} \binom{n-i-1-j, k}{j} x^{(k-1)(n-i-1)-kj+k-i} \\
&= \sum_{i=0}^{k-1} \sum_{j \geqslant 0} \binom{n-i-2-j, k}{j} x^{(k-1)(n-1)-ki-kj} \\
&= \sum_{i=0}^{k-1} \sum_{j \geqslant i} \binom{n-2-j, k}{j-i} x^{(k-1)(n-1)-kj} \\
&= \sum_{i=0}^{k-1} \sum_{j \geqslant 0} \binom{n-2-j, k}{j-i} x^{(k-1)(n-1)-kj} \\
&= \sum_{j \geqslant 0} x^{(k-1)(n-1)-kj} \sum_{i=0}^{k-1} \binom{n-2-j, k}{j-i} \\
&= \sum_{j \geqslant 0} \binom{n-1-j, k}{j} x^{(k-1)(n-1)-kj} = F_n^k(x),
\end{aligned}$$

which in virtue of (3) completes the proof.

In (Philippou et al., 1983) it was proved that the sequence $(U_n^{\mu}(x))_{n \in \mathbb{N}}$ defined by

$$\begin{aligned}
U_0^k(x) &= 0, \\
U_{n+1}^k(x) &= \sum_{n_1+2n_2+\ldots+kn_k=n} \frac{(n_1+n_2+\ldots+n_k)!}{n_1!n_2!\ldots n_k!} x^{k(n_1+n_2+\ldots+n_k)-n}
\end{aligned} \tag{6}$$

is the sequence of the Fibonacci polynomials of order k. Hence, from Theorem 1 we get the following result.

Corollary 1
The sequences of polynomials defined by (6) and (5) are equal.

Now we turn to the Fibonacci numbers of order $k \in \mathbb{N}_2$, i.e. the sequence defined by

$$\begin{aligned} F_0^k &= 0, \\ F_1^k &= 1, \\ F_n^k &= 2^{n-2}, \qquad n \in \{2, 3, \dots k-1\}, \\ F_n^k &= \sum_{j=1}^{k} F_{n-j}^k, \quad n \geqslant k. \end{aligned}$$

As it was mentioned, this sequence might be obtained by substituting $x = 1$ to the sequence of the Fibonacci polynomials of order k, i.e. $(F_n^k(1))_{n\in\mathbb{N}}$. Therefore we obtain

Corollary 2
Let $k \in \mathbb{N}_2$, then the sequence $(F_n^k)_{n\in\mathbb{N}}$ defined by

$$F_n^k := F_n^k(1) = \sum_{j\geqslant 0} \binom{n-1-j,k}{j}, \; n \in \mathbb{N}$$

is the sequence of the Fibonacci numbers of order k.

Let us mention that the above corollary is the main result of (Belbachir et al., 2008), and it was obtained by different methods. Namely, by the means of the ordinary multinomials and the partial Bell partition polynomials. The Fibonacci numbers of different orders were also considered in (Hoggatt, Bicknell, 1973; Koshy, 2001; Schork, 2008).

We finish with the following remark. Consider the sequence $(F_n^k(x))_{n\in\mathbb{Z}}$ satisfying (2) and, moreover,

$$F_n^k(x) = \sum_{j=1}^{k-1} x^{k-j} F_{n-j}^k(x) \tag{7}$$

for every $n \in \mathbb{Z}$.

Substituting $n = k-1$, $n = k-3$, … $n = 2$ into (7) we get

$$F_{-1}^k(x) = 0, \; F_{-2}^k(x) = 0, \; \dots, F_{-(k-2)}^k(x) = 0,$$

respectively.

For this reason, many authors assume exactly k initial conditions for a sequence of the Fibonacci numbers or polynomials of order k. These conditions may be put in a sequence of k elements as follows $(0, 0, \dots, 0, 1)$.

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