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Prime counting function π*

Abstract. The aim of this paper is to derive new explicit formulas for the function π, where $\pi(x)$ denotes the number of primes not exceeding x. Some justifications and generalisations of the formulas obtained by Willans (1964), Minac (1991) and Kaddoura and Abdul-Nabi (2012) are also obtained.

The inspiration to this paper were known results by C. P. Willans, J. Kaddoura and S. Abdul-Nabi (see Willans, 1964; Kaddoura, Abdul-Nabi, 2012). In this paper we deal with the prime counting function, i.e., the function $\pi(x)$ giving the number of primes less than or equal to a given number x. We recall a few known formulas expressing the function π. We also give some new formulas for $\pi(x)$.

We start with recalling some basic facts and notations. Let $\mathbb{P}$ denote the set of all prime numbers, $[x]$ stand for the integer part of $x \in \mathbb{R}$ and let

$$\mathbb{N}_k := \{k, k+1, k+2, \ldots\},$$

where k is an arbitrary fixed positive integer.

In 1964 C. P. Willans gave the following two formulas

$$\pi(n) = \sum_{j=2}^{n} \left[\cos^2 \pi \frac{(j-1)!+1}{j}\right] \quad \text{for} \quad n \in \mathbb{N}_2, \tag{1}$$

$$\pi(n) = \sum_{j=2}^{n} \frac{\sin^2 \pi \frac{((j-1)!)^2}{j}}{\sin^2 \frac{\pi}{j}} \quad \text{for} \quad n \in \mathbb{N}_2 \quad \text{(Willans, 1964)}. \tag{2}$$

In (Ribenboim, 1991) one may find the following formula discovered by J. Mináč

$$\pi(n) = \sum_{j=2}^{n} \left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j}\right]\right] \quad \text{dla} \quad n \in \mathbb{N}_2. \tag{3}$$

A similar formula was given also in (Kaddoura, Abdul-Nabi, 2012). Let us remark that a different approach to the function $\pi(x)$ may be found in (Lagarias, Miller,

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*Funkcja π zliczająca liczby pierwsze

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Odlyzko, 1985) and (Oliveira e Silva, 2006). For $n \in \mathbb{N} \setminus 2\mathbb{N}$ let $n!!$ denote the product of all positive odd integers less than or equal to n, i.e. $n!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$ and if $n \in 2\mathbb{N}_1$ let $n!!$ be the product of all positive even integers less than or equal to n, i.e. $n!! = 2 \cdot 4 \cdot \ldots \cdot n$. Set also $0!! := 1$.
Furthermore, let $n!^2$ and $n!!^2$ denote $(n!)^2$ and $(n!!)^2$, respectively.

In the sequel we will use the following necessary and sufficient conditions for a positive integer $n \geqslant 2$ to be a prime.

(A) $n \in \mathbb{P} \Leftrightarrow n|((n-1)!+1)$ (Ribenboim, 1991, p. 36),

(B) $n \in \mathbb{P} \Leftrightarrow n|((n-2)!-1)$ (Sierpiński, 1962, p. 41),

(C) $n \in \mathbb{P} \Leftrightarrow n|\left(\left[\frac{n}{2}\right]!^2 + (-1)^{\left[\frac{n}{2}\right]}\right)$ (Górowski, Łomnicki, 2013),

(D) $n \in \mathbb{P} \Leftrightarrow n|\left((n-2)!!^2 + (-1)^{\left[\frac{n}{2}\right]}\right)$ (Górowski, Łomnicki, 2013),

(E) $n \in \mathbb{P} \Leftrightarrow n|\left((n-1)!!^2 + (-1)^{\left[\frac{n}{2}\right]}\right)$ (Górowski, Łomnicki, 2013).

Notice that condition (A) is the famous Willson's theorem and (B) is called the Leibniz's theorem.

We begin by proving the following result.

Theorem 1
If $f\colon \mathbb{N}_2 \to \mathbb{Z}$ is a function such that

$$\forall p \in \mathbb{P}\ \frac{f(p)}{p} \in \mathbb{Z} \quad \textit{and} \quad \forall n \in \mathbb{N}_2 \setminus \mathbb{P}\ \frac{f(n)}{n} \notin \mathbb{Z},$$

then

$$\pi(n) = \sum_{j=2}^{n} \left[\frac{f(j)}{j} - \left[\frac{f(j)-j}{j}\right]\right], \qquad n \in \mathbb{N}_2.$$

Proof. It suffices to show that

1. $\left[\frac{f(j)}{j} - \left[\frac{f(j)-1}{j}\right]\right] = 1$, if $j \in \mathbb{P}$,

2. $\left[\frac{f(j)}{j} - \left[\frac{f(j)-1}{j}\right]\right] = 0$, if $j \in \mathbb{N}_2 \setminus \mathbb{P}$.

Suppose that $j \in \mathbb{P}$. Then $f(j) = k \cdot j$ for some $k \in \mathbb{Z}$ and

$$\frac{f(j)}{j} - \left[\frac{f(j)-1}{j}\right] = \frac{k \cdot j}{j} - \left[\frac{kj-1}{j}\right] = k - \left[k - \frac{1}{j}\right] = k - (k-1) = 1.$$

Now assume that $j \in \mathbb{N}_2 \setminus \mathbb{P}$. Then $f(j) = k \cdot j + r$ for some $k \in \mathbb{Z}$ and $r \in \mathbb{N}$, where $0 < r \leqslant j-1$. Hence

$$\left[\frac{f(j)-1}{j}\right] = \left[k + \frac{r-1}{j}\right] = k$$

and
$$\left[\frac{f(j)}{j} - \left[\frac{f(j)-1}{j}\right]\right] = \left[k + \frac{r}{j} - k\right] = \left[\frac{r}{j}\right] = 0.$$
This completes the proof.

Theorem 2
If $g\colon \mathbb{N}_2 \to \mathbb{R}$ is a function satisfying
$$\forall p \in \mathbb{P}\ \frac{g(p)}{p} \in \mathbb{Z} \quad \textit{and} \quad \forall n \in \mathbb{N}_2 \setminus \mathbb{P}\ \frac{g(n)}{n} \notin \mathbb{Z},$$
then
$$\pi(n) = \sum_{j=2}^{n} \left[\cos^2 \pi \frac{g(j)}{j}\right] \qquad \textit{for } n \in \mathbb{N}_2.$$

Proof. For the proof it is enough to notice that by the definition of g we get
$$\left[\cos^2 \pi \frac{g(j)}{j}\right] = \begin{cases} 1, \text{ if } j \in \mathbb{P}, \\ 0, \text{ if } j \in \mathbb{N}_2 \setminus \mathbb{P}. \end{cases}$$

Theorem 3
If $h\colon \mathbb{N}_2 \to \mathbb{R}$ is a function such that
$$\forall n \in \mathbb{N}_2 \setminus \mathbb{P}\ \frac{h(n)}{n} \in \mathbb{Z} \quad \textit{and} \quad \forall p \in \mathbb{P}\ \exists^1 a \in \{-1,1\} :\ \frac{h(p)+a}{p} \in\ \mathbb{Z},$$
then
$$\pi(n) = \sum_{j=2}^{n} \frac{\sin^2 \pi \frac{h(j)}{j}}{\sin^2 \frac{\pi}{j}}.$$

Proof. Notice that for $j \in \mathbb{N}_2 \setminus \mathbb{P}$ we have $\sin^2 \pi \frac{h(j)}{j} = 0$.
Suppose that $j \in \mathbb{P}$, then
$$\sin \pi \frac{h(j)}{j} = \sin \pi \frac{h(j)+a-a}{j} = \sin \pi \frac{h(j)+a}{j} \cos \pi \frac{a}{j} - \cos \pi \frac{h(j)+a}{j} \sin \pi \frac{a}{j},$$
where $a \in \{-1,1\}$ satisfies $\frac{h(j)+a}{j} \in \mathbb{Z}$. Thus we obtain $\sin^2 \pi \frac{h(j)}{j} = \sin^2 \frac{\pi}{j}$ and $\frac{\sin^2 \pi \frac{h(j)}{j}}{\sin^2 \frac{\pi}{j}} = 1$ for $j \in \mathbb{P}$ and the proof is completed.

Corollary 1 (Corollary to Theorem 1)
Let the function f be given by one of the following formulas:
$$\begin{aligned} &f(n) = (n-1)! + 1, && f(n) = (n-2)! - 1, \\ &f(n) = \left[\tfrac{n}{2}\right]!^2 + (-1)^{[\frac{n}{2}]}, && f(n) = (n-2)!!^2 + (-1)^{[\frac{n}{2}]}, \\ &f(n) = (n-1)!!^2 + (-1)^{[\frac{n}{2}]}. \end{aligned} \tag{4}$$
Then by Theorem 1, in view of (A), (B), (C), (D), (E) *we obtain five formulas for the function π, including, given by J. Mináč, formula* (3).

Corollary 2 (Corollary to Theorem 2)
Let $g(n) = f(n)$, $n \in \mathbb{N}_2$, where f is the function defined by one of the formulas in (4)*. Then by Theorem 2, in view of* (A), (B), (C), (D), (E) *we obtain five formulas for the function π, including* (1) *– given by C. P. Willans.*

Corollary 3 (Corollary to Theorem 3)
Let h be the function given by one of the following

$$h(n) = (n-1)!^2, \quad h(n) = (n-2)!^2, \quad h(n) = \left[\frac{n}{2}\right]!^2.$$

Then from Theorem 3 in virtue of (A), (B), (C) *we get three formulas for π, including, given by C. P. Willans, formula* (2).

Now we prove

Theorem 4
The function π may by expressed by each of the following formulas:

(i) $$\pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{\cos^2 \frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1}}{\cos^2 \frac{\pi}{2(2j+1)}} \quad \textit{for} \quad n \in \mathbb{N}_2,$$

(ii) $$\pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{|\cos \frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1}|}{\cos \frac{\pi}{2(2j+1)}} \quad \textit{for} \quad n \in \mathbb{N}_2.$$

Proof. Notice that for $n = 2$ we have $\pi(2) = 1$. Let $n > 2$. It suffices to show that

$$\cos \frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1} = 0, \quad \text{if} \quad 2j+1 \in \mathbb{N}_2 \setminus (2\mathbb{N} \cup \mathbb{P})$$

and

$$\left|\cos \frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1}\right| = \cos \frac{\pi}{2(2j+1)}, \quad \text{if} \quad 2j+1 \in \mathbb{P} \setminus \{2\}.$$

Fix $j \in \mathbb{N}$ such that $2j+1 \in \mathbb{N}_2 \setminus (2\mathbb{N} \cup \mathbb{P})$, hence $(2j+1) | (2j-1)!!^2$. Moreover, $(2j-1)!!^2 = l(2j+1)$, where l is a positive odd integer. It follows that

$$\cos \frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1} = 0.$$

Now let $j \in \mathbb{N}$ be such that $2j+1 \in \mathbb{P} \setminus \{2\}$. By (D) we obtain

$$(2j-1)!!^2 + (-1)^j = 2k(2j+1),$$

where k is a positive integer and

$$\cos \frac{\pi}{2}\frac{(2j-1)!!^2 + (-1)^j - (-1)^j}{2j+1} = \cos\left(\frac{\pi}{2} \cdot 2k\right) \cos \frac{\pi(-1)^j}{2(2j+1)} + \sin\left(\frac{\pi}{2} \cdot 2k\right) \sin \frac{\pi(-1)^j}{2(2j+1)}.$$

This yields $\left|\cos\frac{\pi}{2}\frac{(2j-1)!!^2}{2j+1}\right| = \cos\frac{\pi}{2(2j+1)}$.

The following result may be proved similarly as Theorem 4.

THEOREM 5
If $l(n) = (n-1)!$ *or* $l(n) = (n-1)!!^2$ *for* $n \in \mathbb{N}_2$, *then*

$$\text{(i)}\ \ \pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{\sin^2\frac{\pi}{2}\frac{l(2j+1)}{2j+1}}{\cos^2\frac{\pi}{2(2j+1)}} \quad \text{for} \quad n \in \mathbb{N}_2,$$

$$\text{(ii)}\ \ \pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{|\sin\frac{\pi}{2}\frac{l(2j+1)}{2j+1}|}{\cos\frac{\pi}{2(2j+1)}} \quad \text{for} \quad n \in \mathbb{N}_2.$$

Using the same reasoning as in the proofs of Theorems 3 and 4 one may show

THEOREM 6
Let $k\colon \mathbb{N}_2 \to \mathbb{R}$ *be a function satisfying*

$$\forall n \in \mathbb{N} \setminus (2\mathbb{N} \cup \mathbb{P})\ \frac{k(n)}{n} \in \mathbb{Z} \quad \text{and} \quad \forall p \in \mathbb{P} \setminus \{2\}\ \exists a \in \{-1, 1\} :\ \frac{k(p)+a}{p} \in \mathbb{Z},$$

then

$$\text{(i)}\ \ \pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{\sin^2\pi\frac{k(2j+1)}{2j+1}}{\sin^2\frac{\pi}{2j+1}} \quad \text{for } n \in \mathbb{N}_2,$$

$$\text{(ii)}\ \ \pi(n) = 1 + \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \frac{|\sin\pi\frac{k(2j+1)}{2j+1}|}{\sin\frac{\pi}{2j+1}} \quad \text{for } n \in \mathbb{N}_2.$$

COROLLARY 4 (COROLLARY TO THEOREM 6)
Let k *be the function given by one of the following formulas:* $k(n) = (n-1)!$, $k(n) = (n-2)!$, $k(n) = \left[\frac{n}{2}\right]!^2$, $k(n) = (n-2)!!^2$, $k(n) = (n-1)!^2$, $k(n) = (n-2)!^2$, $k(n) = (n-1)!!^2$. *Then by Theorem 6 and in view of conditions* (A), (B), (C), (D), (E) *we obtain other formulas for the function* π.

The following formula for the n-th prime was given in (Willans, 1964)

$$p_n = 1 + \sum_{m=1}^{2^n} \left[\left(\frac{n}{1+\pi(m)}\right)^{\frac{1}{n}}\right] \text{(Willans, 1964).} \tag{5}$$

Let π be the function given by the formulas obtained by Corollaries 1, 2, 3 and by conditions (i) and (ii) of Theorems 4, 5. Put moreover $\pi(1) = 0$. Then by (5) we get numerous formulas for the n-th prime.

Bibliography

Górowski, J., Łomnicki, A.: 2013, Around the Wilson's theorem, *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* **V**, 51-56.

Kaddoura, J., Abdul-Nabi, S.: 2012, On formula to compute primes and the n^{th} prime, *Applied Math. Sciences* **6**(76), 3751-3757.

Lagarias, J. C., Miller, V. S., Odlyzko, A. M.: 1985, Computing $\pi(x)$: the Meissel-Lehmer method, *Math. Comp.* **44**(170), 537-560.

Oliveira e Silva, T.: 2006, Computing $\pi(x)$: the combinatorial method, *Revista do Detua* **4**(6), 759-768.

Ribenboim, P.: 1991, *The little book of big primes*, Springer Verlag, New York.

Sierpiński, W.: 1962, *Co wiemy a czego nie wiemy o liczbach pierwszych*, PZWS, Warszawa.

Willans, C. P.: 1964, On formulae for the n-th prime, *Math. Gaz.* **48**, 413-415.

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